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Analytical solution of the spherical indentation problem for a half-space with gradients with the depth elastic properties

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Abstract

The contact problem for the impression of spherical indenter into a non-homogeneous (both layered and functionally graded) elastic half-space is considered. Analytical methods for solving this problem have been developed. It is assumed that the Lame coefficients vary arbitrarily with the half-space depth. The problem is reduced to dual integral equations. The developed methods make it possible to find the analytical asymptotically exact problem solution, suitable for a PC. The influence of the Lame coefficients variation upon the contact stresses and size of the contact zone with different radius of indenter as well as values of the impressing forces are studied. The effect of the non-homogeneity is examined. The developed method allows to construct analytical solutions with presupposed accuracy and gives the opportunity to do multiparametric and qualitative investigations of the problem with minimal computation time expenditure. © 2002 Elsevier Science Ltd. All rights reserved.

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1. Introduction

The analysis of the interaction between a smooth rigid sphere and an elastic half-space is a fundamental problem in contact mechanics. The solution of this problem was developed by Hertz (1881) and currently the methods of the integral equation solution obtained from the Hertzian contact problem for a homogeneous half-space are well known; e.g., Johnson (1985), Gladwell (1980).

El-Sherbiney and Halling (1976) extended the classical Hertzian arguments to the system contact, where one surface is covered with a layer of a homogeneous material possessing elastic properties different from those of the substructure.

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The problem was reduced to an integral equation. This approximate solution was obtained by El-Sherbiney and Halling (1976) by two different methods. One of these methods described by Popov (1962) is efficient only for very thin layers lying on the rigid non-deformable substrate, in particular when layer thickness H is less than the radius of contact zone $a(\lambda = H/a < 0.5)$. The second method used Vorovich and Ustinov (1959) is applicable in the case when layer thickness is as follows: $\lambda > 1.7$. Both of these methods with slight modifications are applicable in the case when one of contacting bodies is a layer lying on an elastic half-space.

The same problem was investigated by Potelejko and Filippov (1967). The solution of the integral equation was obtained numerically.

Different aspects of generalized statements of the Hertzian problem have been considered in the recent works of Kral et al. (1993, 1995), Moutmitounet et al. (1993), Yingzhi and Hills (1991), and Zeng et al. (1992).

In the work of Suresh et al. (1997) new finite element computations were reported on the micromechanics of the penetration of a spherical indenter into a graded material and on the evolution of stress, strain and displacement fields around the indenter. The numerical simulations were compared with the analytical results.

In the present paper the Hertzian contact problem for both layered and functionally graded half-space is studied analytically. We presume that the variation of the Lame coefficients with depth has general nature (arbitrary continuous or piecewise continuous functions of depth). We assume elastic properties of a half-space to become stable with depth. That is to say, it can be imagined as a non-homogeneous layer of thickness H, which is completely coupled with a homogeneous half-space. Difficulties arise at the very first stage when we use the method of integral transforms for the solution of this problem, because it is necessary to solve the two-point boundary problem for the system of ordinary differential equations with the variable coefficients in order to construct the transform of the integral equation kernel. To construct it we use an effective special scheme of the method developed by Babeshko et al. (1987).

The numerically constructed kernel transform is approximated by an analytical expression of a special type, so that it becomes to obtain a closed analytical solution of the approximate integral equation Aizikovich and Aleksandrov (1984). It is shown that the resulting approximate solution is the bilateral asymptotically exact problem solution for small and large values of a characteristic geometric parameter.

The accuracy of the obtained solution is studied in this paper.

2. Formulation of the problem

A non-deformable spherical indenter is impressed into surface Γ of a non-homogeneous elastic half-space Ω by force P (Fig. 1). Cylindrical (r, φ, z) coordinates relate to the half-space. It is assumed that all deformations are elastic and the size of the contact zone a is small respectively to the radius of the sphere R, while no friction force exists between the indenter and the surface of the half-space. The spherical indenter surface in the vicinity of the original point of contact is approximated by a quadratic shape $z = \psi(r) = \beta r^2$. The half-space is not loaded outside the indenter. Under the action of the force P, the indenter moves a distance χ along the z-axis.

The Lame coefficients $\Lambda(z)$ and $M(z)$ in the half-space very generally, that is, they are arbitrarily continuous or piecewise continuous functions of the depth z.

$$\begin{aligned} &\Lambda(z) = \Lambda_0(z), \quad M(z) = M_0(z), \quad -H \leqslant z \leqslant 0 \\ &\Lambda = \Lambda_1 = \Lambda_0(-H), \quad M = M_1 = M_0(-H), \quad -\infty < z < -H \end{aligned} \tag{1}$$

Let us suppose that the following conditions are satisfied

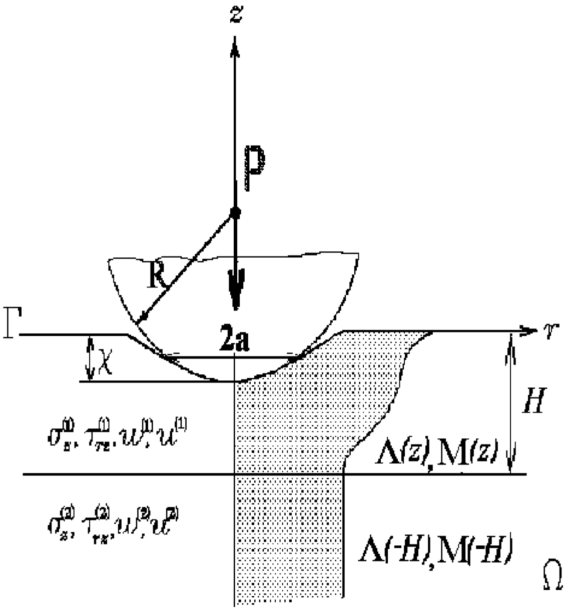

Fig. 1. Scheme of indentation test.

$$\min_{z\in(-\infty;0]} \Lambda(z) \geqslant c_1 > 0, \qquad \max_{z\in(-\infty;0]} \Lambda(z) \leqslant c_2 < \infty$$
$$\min_{z\in(-\infty;0]} M(z) \geqslant c_3 > 0, \qquad \max_{z\in(-\infty;0]} M(z) \leqslant c_4 < \infty \tag{2}$$

where c_1, c_2, c_3, c_4 are certain constants.

Under the above assumption, the boundary conditions have the form

$$z = 0, \quad \tau_{zr} = \tau_{z\varphi} = 0, \quad \begin{cases} \sigma_z = 0, & r > a \\ w = \chi(r) = \chi - \psi(r), & 0 \leqslant r \leqslant a \end{cases} \tag{3}$$

Here w is the displacement along the z-axis, τ_{zr}, $\tau_{z\varphi}$, σ_z are radial, tangential and normal stresses, respectively.

For $z = -H$, the following conditions for the stress and strain must be performed:

$$\sigma_z^{(1)} = \sigma_z^{(2)}, \quad \tau_{rz}^{(1)} = \tau_{rz}^{(2)}, \quad w^{(1)} = w^{(2)}, \quad u^{(1)} = u^{(2)} \tag{4}$$

Here u is the radial displacement, $\tau_{rz}^{(1)}$, $\sigma_z^{(1)}$, $w^{(1)}$, $u^{(1)}$ are stresses and displacements of the coating and $\tau_{rz}^{(2)}$, $\sigma_z^{(2)}$, $w^{(2)}$, $u^{(2)}$ are stresses and displacements of the substrate, which is a homogeneous half-space.

The stress and strain vanish for $(r, -z) \to \infty$. It is required to determine the size of the contact zone a, the distribution of the contact normal stresses $q(r)$ under the indenter

$$\sigma_z(r, 0) = -q(r), \quad 0 \leqslant r \leqslant a \tag{5}$$

and to find the impressing force P. Here the problem of finding contact stress distribution of the function $q(r)$ is fundamental because after the determination of $q(r)$, the force P acting on the indenter can be found from the condition of equilibrium of the indenter:

$$P = 2\pi \int_0^a q(\rho)\rho\,\mathrm{d}\rho \tag{6}$$

In the present case, the edge of the indenter does not cut into the surface of the half-space. Consequently, this relation must hold:

$$q(a) = 0 \tag{7}$$

It is used for the determination of the contact region half-width a and imposes a certain restriction on function $q(r)$.

It should be noted also that as there is no adhesion between the indenter and the half-space surface, the relation $q(r) \geqslant 0$ must hold for all $r \leqslant a$ for the correct statement of a problem.

3. Reduction of the problem to the dual integral equation

We use static equilibrium equations of the theory of elasticity written in the displacements in the case of axially symmetrical deformation for the cylindrical coordinate system:

$$\begin{gathered}
r\frac{\partial\sigma_z}{\partial z} + \frac{\partial}{\partial r}(r\tau_{rz}) = 0 \\
\frac{\partial}{\partial r}(r\sigma_r) - \sigma_\varphi + r\frac{\partial\tau_{rz}}{\partial z} = 0 \\
\sigma_z = 2M\frac{\partial w}{\partial z} + \Lambda\theta, \qquad \sigma_r = 2M\frac{\partial u}{\partial r} + \Lambda\theta \\
\sigma_\varphi = 2M\frac{u}{r} + \Lambda\theta, \qquad \tau_{rz} = M\left(\frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}\right) \\
\theta = \frac{\partial w}{\partial z} + \frac{\partial u}{\partial r} + \frac{u}{r}
\end{gathered} \tag{8}$$

These equations can be presented in the next form, Ter-Mkrtch'ian (1961)

$$\begin{gathered}
M\left(\nabla^2 u + \frac{\partial\theta}{\partial r} - \frac{u}{r^2}\right) + \frac{\partial M}{\partial z}\left(\frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}\right) + \frac{\partial}{\partial r}(\Lambda\theta) = 0 \\
M\left(\nabla^2 w + \frac{\partial\theta}{\partial z}\right) + 2\frac{\partial M}{\partial z}\frac{\partial w}{\partial z} + \frac{\partial}{\partial z}(\Lambda\theta) = 0 \\
\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2}
\end{gathered} \tag{9}$$

We present radial displacement u, and vertical displacement w, in the Hankel integral form

$$\begin{gathered}
u_i(r,z) = -\int_0^\infty U_i(\gamma,z)J_1(\gamma r)\gamma\,\mathrm{d}\gamma \\
w_i(r,z) = \int_0^\infty W_i(\gamma,z)J_0(\gamma r)\gamma\,\mathrm{d}\gamma \quad (i = 1,2)
\end{gathered} \tag{10}$$

here subscript $i = 1$ corresponds to the coating and $i = 2$ corresponds to the substrate (a homogeneous half-space), J_1 is the Bessel function of the first order, J_0 is the Bessel function of the zero order.

Eq. (9) with (10) can be presented in the form

$$\begin{gathered}
MU_i'' + \gamma(M+\Lambda)W_i' - \gamma^2(2M+\Lambda)U_i + M'U_i' + \gamma M'W_i = 0 \\
(2M+\Lambda)W_i'' - \gamma(M+\Lambda)U_i' - \gamma^2 MW_i + \left(2M' + \Lambda'\right)W_i' - \gamma\Lambda' U_i = 0 \\
i = 1, \quad z \in [-H;0]; \quad i = 2, \quad z \in (-\infty;-H)
\end{gathered} \tag{11}$$

Here the prime sign indicates the differential with respect to z.

We introduce the auxiliary functions

$$\begin{aligned} &U_i^*(\gamma,z) = -\Theta(0)U_i(\gamma,z)\gamma Q^{-1}(\gamma) \\ &W_i^*(\gamma,z) = -\Theta(0)W_i(\gamma,z)\gamma Q^{-1}(\gamma) \quad (i=1,2) \\ &\Theta(0) = 2M(0)(\Lambda(0)+M(0))(\Lambda(0)+2M(0))^{-1}, \quad M(0)\neq 0 \end{aligned} \tag{12}$$

$$Q(\gamma) = \int_0^a q(\rho)J_0(\gamma\rho)\rho\,\mathrm{d}\rho \tag{13}$$

$$q(\rho) = \int_0^\infty Q(\gamma)J_0(\gamma\rho)\gamma\,\mathrm{d}\gamma \tag{14}$$

where $Q(\gamma)$ is the Hankel transform of the unknown function $q(r)$. To reduce the mixed problem to an integral equation it is necessary to construct the function $W_1^*(\gamma,0)$.

4. Construction of the integral equation kernel function

Let us consider auxiliary problems with the following boundary conditions (prescribed normal stress):

$$z=0, \quad \tau_{rz}=\tau_{z\varphi}=0, \quad \sigma_2(r,0) = \begin{cases} 0, & r>a \\ -q(r), & 0\leqslant r\leqslant a \end{cases} \tag{15}$$

Conditions (4) are satisfied accordingly at $z=-H$.

We introduce the notation

$$v_1 = U^*, \quad v_2 = U^{*'}, \quad v_3 = W^*, \quad v_4 = W^{*'} \tag{16}$$

Here the prime indicates the differentiation with respect to z. We rewrite the system (11) in the matrix form:

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}y} = \mathbf{A}\vec{v}$$

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \gamma^2\frac{2M+\Lambda}{M} & -\frac{M'}{M} & -\gamma\frac{M'}{M} & -\gamma\frac{M+\Lambda}{M} \\ 0 & 0 & 0 & 1 \\ \gamma\frac{\Lambda'}{2M+\Lambda} & \gamma\frac{M+\Lambda}{2M+\Lambda} & \gamma^2\frac{M}{2M+\Lambda} & -\frac{2M'+\Lambda'}{2M+\Lambda} \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \tag{17}$$

The general solution of the system (11) with the condition $\Lambda' = M' = 0$ $(M\neq 0)$ has the form

$$\begin{aligned} &U_2^*(\gamma,z) = (d_1+\gamma z d_2)\mathrm{e}^{\gamma z} \\ &W_2^*(\gamma,z) = (d_1-\kappa d_2-\gamma z d_2)\mathrm{e}^{\gamma z} \\ &\kappa = 3-4\nu = (\Lambda+3M)/(\Lambda+M) \end{aligned} \tag{18}$$

where d_1, d_2 are unknown constants, ν is Poisson's ratio. The solution, $\vec{v}(\gamma,z)$ of (17) is constructed by the method of simulating functions of Babeshko et al. (1987). According to the general scheme of the method of simulating functions for the Eq. (17), we extend the solution of the differential equation with the constant coefficients for $z<-H$, on the solution of the differential equation with the varying coefficients for $z\in[-H;0]$ by means of multiplying constants on some unknown functions in the former solution.

$$\vec{v}(\gamma, z) = \left[\sum_{i=1}^{2} d_i(\gamma)\vec{a}_i(\gamma, z)\right] e^{\gamma z} \tag{19}$$

Here we find vectors $\vec{a}_1(\gamma, z)$, $\vec{a}_2(\gamma, z)$ from the following Cauchy problems:

$$\frac{d\vec{a}_i}{dz} = \mathbf{A}\vec{a}_i - \gamma\vec{a}_i, \quad z \in [-H; 0] \ (i = 1, 2) \tag{20}$$

with the initial conditions

$$\begin{aligned} &\vec{a}_1(\gamma, -H) = (1, \gamma, 1\gamma), \\ &\vec{a}_2(\gamma, -H) = \left(\gamma z, \gamma + \gamma^2 z, -\kappa + \gamma z, -\kappa\gamma + \gamma + \gamma^2 z\right)\big|_{z=-H} \end{aligned} \tag{21}$$

Vectors of the initial conditions, $\vec{a}_1(\gamma, -H)$ and $\vec{a}_2(\gamma, -H)$ defined from (19) under the condition that, $\vec{v}(\gamma, z)$ should be the same as the solution of (18) for all $z < -H$.

Functions $d_1(\gamma)$, $d_2(\gamma)$ are determined from the conditions (15)

$$\begin{aligned} &\sigma_z(r, 0) = -q(r), \quad r \leqslant a \Rightarrow \sum_{i=1}^{2} d_i(\gamma)\left[-\Lambda(0)\gamma a_i^1(\gamma, 0) + (\Lambda(0) + 2M(0))a_i^4(\gamma, 0)\right] = \Theta(0)\gamma \\ &\tau_{rz}(r, 0) = 0, \quad \forall r \Rightarrow \sum_{i=1}^{2} d_i(\gamma)\left[a_i^2(\gamma, 0) + \gamma a_i^3(\gamma, 0)\right] = 0 \end{aligned} \tag{22}$$

Here $a_i^k(\gamma, z)$ denotes the kth component of the vector $\vec{a}_i(\gamma, z)$ and the sums are taken over $i = 1, 2$ ($k = 1$–4). The system (22) is uniquely solvable if its determinant is not equal to zero.

From (5) we finally obtain the expression for the kernel transform of the integral equation, the function $L^*(\gamma)$:

$$L^*(\gamma) = W_1^*(\gamma, 0) = \sum_{i=1}^{2} d_i(\gamma) a_i^3(\gamma, 0) \tag{23}$$

After determining $W_1^*(\gamma, 0)$, we apply the condition (3) and write it in the next form:

$$\Theta^{-1}(0) \int_0^\infty W_1^*(\gamma, 0) Q(\gamma) J_0(\gamma r)\, d\gamma = \chi - \psi(r), \quad 0 \leqslant r \leqslant a \tag{24}$$

Using the conditions (13) and (14)

$$\int_0^a q(\rho)\rho\, d\rho \int_0^\infty W_1^*(\gamma, 0) J_0(\gamma r) J_0(\gamma\rho)\, d\gamma = \Theta(0)(\chi - \psi(r)), \quad 0 \leqslant r \leqslant a \tag{25}$$

we make the change of variables and note

$$\begin{aligned} &\gamma H = \varsigma, \quad \lambda = \frac{H}{a}, \quad \tilde{r} = \frac{r}{a}, \quad \tilde{\varrho} = \frac{\varrho}{a} \\ &W_1^*(\gamma, 0) \equiv L^*(\gamma) \equiv L^*\left(\frac{\varsigma}{H}\right) = L(\varsigma), \quad q(\tilde{\rho}a) = t(\tilde{\rho}) \\ &f(\tilde{r}) = \delta - \frac{\psi(\tilde{r}a)}{a}, \quad \delta = \frac{\chi}{a}, \quad 0 \leqslant \tilde{r} \leqslant 1 \end{aligned} \tag{26}$$

Let us rewrite (25)

$$\frac{1}{\lambda} \int_0^1 t(\tilde{\rho})\tilde{\rho}\, d\tilde{\rho} \int_0^\infty L(\varsigma) J_0\left(\frac{\tilde{r}\varsigma}{\lambda}\right) J_0\left(\frac{\tilde{\rho}\varsigma}{\lambda}\right) d\varsigma = \Theta(0) f(\tilde{r}), \quad 0 \leqslant \tilde{r} \leqslant 1 \tag{27}$$

Let us denote $\varsigma/\lambda = \alpha = \gamma a$

$$\int_0^1 t(\tilde{\rho})\tilde{\rho}\,d\tilde{\rho}\int_0^\infty L(\lambda\alpha)J_0(\alpha\tilde{r})J_0(\alpha\tilde{\rho})\,d\alpha = \Theta(0)f(\tilde{r}), \quad 0 \leqslant \tilde{r} \leqslant 1 \tag{28}$$

It should be noted that in the following we consider dimensionless variables and omit tildes under them for short.

Since the problem formulated for the case of the spherical indenter, $\psi = \beta r^2$ is reduced to the following type of the dual integral equations

$$\begin{cases} \int_0^\infty T(\alpha)L(\lambda\alpha)J_0(\alpha r)\,d\alpha = \Theta(0)f(r), & 0 \leqslant r \leqslant 1 \\ \int_0^\infty T(\alpha)J_0(\alpha r)\alpha\,d\alpha = 0, & r > 1 \end{cases} \tag{29}$$

$$T(\alpha) = \int_0^1 t(\rho)J_0(\alpha\rho)\rho\,d\rho \tag{30}$$

5. The general properties of the kernel transforms of the integral equation of the problem

When the following conditions are satisfied

$$\min_{z\in(-\infty;0]}\Theta(z) \geqslant c_1 > 0, \quad \max_{z\in(-\infty;0]}\Theta(z) \leqslant c_2 < \infty, \quad \lim_{z\to-\infty}\Theta(z) = \text{const}$$
$$\Theta(z) = 2M(z)\frac{\Lambda(z)+M(z)}{\Lambda(z)+2M(z)} \tag{31}$$

it can be demonstrated (Aizikovich and Aleksandrov, 1982), that the kernel transform $L(\gamma)$ has the following properties

$$L(\gamma) = p_1 + p_2\gamma + p_3\gamma^2 + O(\gamma^3), \quad \gamma \to 0 \tag{32}$$

$$L(\gamma) = 1 + p_4\gamma^{-1} + p_5\gamma^{-2} + p_6\gamma^{-3} + O(\gamma^{-4}), \quad \gamma \to \infty \tag{33}$$

$$p_1 = \Theta(0)\Theta^{-1}(-H) \tag{34}$$

where p_2, p_3, p_4, p_5, p_6 are some constants.

For multilayer media the property of a compliance function similar to (34) was noticed by Privarnikov (1973).

The property (34) means that the value $L(0)$ for the problem under consideration is independent of the way in which elastic moduli vary in the half-space from $z = 0$ to $z \to -\infty$ and it is determined only by their values for $z = 0$ and $z \to -\infty$. Graphically it looks as follows: if the set of curves describing the certain laws of the elastic moduli variation with depth have identical values on the surface of the half-space and as $z \to -\infty$, then the graphs of corresponding transforms $L(\gamma)$ of the problem will issue from a common point $L(0) = p_1$ and converge at one point $L(\infty) = 1$.

Taking into consideration that the Lame coefficients Λ and M (M is sometimes denoted by G and named the shear modulus) are connected with the Young's modulus E and the Poisson's ratio v by the relations

$$M = G = \frac{E}{2(1+v)}, \quad \Lambda = \frac{Ev}{(1+v)(1-2v)} = \frac{2Gv}{1-2v}$$
$$E = \frac{M(3\Lambda+2M)}{\Lambda+M}, \quad v = \frac{\Lambda}{2(\Lambda+M)}, \quad \Theta = \frac{G}{1-v} = \frac{E}{2(1-v^2)}$$

5.1. The certain auxiliary theorems concerning their analytical approximation

Let us introduce the following definitions:

Definition 1. The function $L(\gamma)$ belongs to the class $\mathbf{\Pi}_N$ if it has the form

$$L_N(\gamma) = \prod_{i=1}^{N} (\gamma^2 + A_i^2)(\gamma^2 + B_i^2)^{-1}, \quad (B_i - B_k)(A_i - A_k) \neq 0, \quad i \neq k \tag{35}$$

Here A_i, B_i $(i = 1, 2, \ldots, N)$ are certain complex constants.

Definition 2. The function $L(\gamma)$ belongs to the class $\mathbf{\Sigma}_M$ if it has the form

$$L_M^{\Sigma}(\gamma) = \sum_{k=1}^{M} C_k \gamma (\gamma^2 + D_k^2)^{-1}$$

Here C_k are certain real constants, and B_k $(k = 1, 2, \ldots, M)$ are certain complex constants.

Definition 3. The function $L(\gamma)$ belongs to the class $\boldsymbol{S}_{N,M}$ if it has the form

$$L(\gamma) = L_N(\gamma) + L_M^{\Sigma}(\gamma) \tag{36}$$

We show that the expressions of the form (36) can approximate $L(\gamma)$ with the properties (32) and (33) using the following lemma (Akhiezer, 1965; Babeshko, 1972):

Lemma 1. *Let an even real function $\varphi(\gamma)$ be continuous on the whole real axis and vanish at infinity, then it can be approximated in $\mathbf{C}(-\infty, \infty)$ by a series of functions of the form*

$$\varphi_k = (\gamma^2 + D_k^2)^{-1}$$

Theorem 1. *Provided that if the function $L(\gamma)$ possesses the properties* (32) *and* (33), *it allows approximation by the expressions of the form* (36).

Proof. We select constants A_i and B_i $(i = 1, 2, \ldots, N)$ in (35) such that

$$\prod_{i=1}^{N} (A_i^2 B_i^{-2}) = p_1 \tag{37}$$

We consider the function

$$L_{\Sigma}(\gamma) = (L(\gamma) - L_N(\gamma))\gamma^{-1} \tag{38}$$

On the basis of properties (32) and (33) and condition (37), it follows that $L_{\Sigma}(\gamma)$ satisfies the condition of Lemma 1. This means that the following representation holds

$$L_{\Sigma}(\gamma) = \sum_{k=1}^{\infty} C_k (\gamma^2 + D_k^2)^{-1} \tag{39}$$

Or from the conditions (38) and (39),

$$L(\gamma) = L_N(\gamma) + \gamma \sum_{k=1}^{M} C_k (\gamma^2 + D_k^2)^{-1} \quad \square \tag{40}$$

For a numerical realization the improving approximation of $L(\gamma)$ by functions of the class $\mathbf{\Pi}_N$ can be achieved successfully by using the following algorithm.

We map function $L(\gamma)$ by mapping $u = \gamma^2/(\gamma^2 + C^2)$ from interval $(0;\infty)$ into segment $(0;1)$ $(\gamma = C\sqrt{u(u-1)^{-1}})$. Here C is a positive constant, which should be selected to build the optimal approximation of function $L(\gamma)$. As initial value C can be taken

$$C = \gamma^*, \quad \text{where } \gamma^* \text{ such as } L(\gamma^*) = \frac{1}{2}\left(\max_{\gamma\in[0;\infty)} L(\gamma) + \min_{\gamma\in[0;\infty)} L(\gamma)\right)$$

Here C is a parameter of mapping, which moves the point $\gamma = C$ of axes $(0;\infty)$ into the point $u = 1/2$ of segment $(0;1)$.

We approximate the functions $\sqrt{L(\gamma)}$ and $\sqrt{L^{-1}(\gamma)}$ on segment $(0;1)$ by Nth order Bernstein's polynomials (or by Chebyshev's nodes), and thus obtain

$$\sqrt{L_N(\gamma)} = \sum_{i=0}^{N} a_i u^i, \qquad \sqrt{L_N^{-1}(\gamma)} = \sum_{i=0}^{N} b_i u^i \tag{41}$$

here a_i, b_i are coefficients of Bernstein's polynomials which can be defined as follows. If $f(x)$ is a continuous function determined on segment $(0;1)$ then Bernstein's polynomial $B_N(x)$ for this function has the form according to Goncharov (1934),

$$B_N(x) = \sum_{m=0}^{N} f\left(\frac{m}{N}\right) C_N^m x^m (1-x)^{N-m}$$

where C_N^m are the binomial coefficients.

Then,

$$\sqrt{L_N(\gamma)} = \left(\sum_{i=0}^{N} a_i^* \gamma^{2i}\right)(\gamma^2 + C^2)^{-N}, \qquad \sqrt{L_N^{-1}(\gamma)} = \left(\sum_{i=0}^{N} b_i^* \gamma^{2i}\right)(\gamma^2 + C^2)^{-N} \tag{42}$$

where coefficients a_i^*, b_i^* are defined from (41) after the change of variable $u = \gamma^2/(\gamma^2 + C^2)$.

$$L_N(\gamma) = \frac{\sqrt{L_N(\gamma)}}{\sqrt{L_N^{-1}(\gamma)}} = \left(\sum_{i=0}^{N} a_i^* \gamma^{2i}\right)\left(\sum_{i=0}^{N} b_i^* \gamma^{2i}\right)^{-1} \tag{43}$$

For each non-homogeneity law, the parameter C is selected separately in order to $L_N(\gamma)$ will approximate the function $L(\gamma)$ more exactly for given N.

By determining the roots of the numerator and denominator in (43), we find the desired values of A_i, B_i $(i = 1, 2, \dots, N)$. Such method permits avoiding the presence of an N-triple root in the denominator of the approximation found.

5.2. Numerical results

The analysis will be carried out for the typical kinds of the layered and functional graded models.

For such models, we suppose the constant Poisson's ratio, $v = 1/3$, and the Young's modulus varies with depth in accordance with the relation

$$E(z) = \begin{cases} E_c^i = E_0 f_i(z), & -H \leqslant z \leqslant 0 \\ E_0, & z < -H \end{cases} \tag{44}$$

$i = 1, 2, \dots, 6$

$$f_1(z) = 3.5, \qquad f_2(z) = \frac{1}{3.5}$$
$$f_3(z) = 3.5 + 2.5\frac{z}{H}, \qquad f_4(z) = \frac{1}{3.5} - \frac{2.5}{3.5}\frac{z}{H}$$
$$f_5(z) = 1 + \frac{2.5}{3.5}\sin\left(\frac{z\pi}{H}\right), \qquad f_6(z) = 1 - 2.5\sin\left(\frac{z\pi}{H}\right)$$

Fig. 2 shows the non-homogeneity laws $f_j(z')$, $j = 0, 1, \ldots, 6$ described above. Here and below $z' = z/H$. Figs. 3–9 show the graphs of the kernels $L(\gamma)$ (curve '1') of the integral equations constructed numerically for the appropriate non-homogeneity laws. On the Figs. 3–9, the curves '2' show the approximation error of the kernel transforms which corresponds to (40) and equals $L(\gamma) - L_N(\gamma)$, for $N = 10$.

Below it is an example of the calculated coefficients A_k, B_k, $k = 1, \ldots, N$ for the non-homogeneity laws $f_1(z)$, $f_2(z)$, in the form $x + iy$, where x is the real part and y is the imaginary one.

Law f_j, $j = 1, 2$	Coefficients, A_k, $k = 1, \ldots, 10$	Values A_k	Coefficients B_k, $k = 1, \ldots, 10$	Values B_k
$f_1(z)$	$A_1 =$	0.842 + 0.217i	$B_1 =$	0.749 + 0.000i
	$A_2 =$	0.842 − 0.217i	$B_2 =$	1.231 + 0.111i
	$A_3 =$	1.329 + 0.068i	$B_3 =$	1.231 − 0.111i
	$A_4 =$	1.329 − 0.068i	$B_4 =$	0.851 + 0.372i
	$A_5 =$	0.992 − 0.611i	$B_5 =$	0.851 − 0.372i
	$A_6 =$	0.992 + 0.611i	$B_6 =$	2.482 + 0.000i
	$A_7 =$	1.673 + 1.103i	$B_7 =$	1.116 + 0.831i
	$A_8 =$	1.673 − 1.103i	$B_8 =$	1.116 − 0.831i
	$A_9 =$	2.395 + 0.474i	$B_9 =$	2.102 + 0.742i
	$A_{10} =$	2.395 − 0.474i	$B_{10} =$	2.102 − 0.742i
$f_2(z)$	$A_1 =$	0.642 + 0.000i	$B_1 =$	2.310 + 0.000i
	$A_2 =$	1.244 + 0.000i	$B_2 =$	1.069 − 0.544i
	$A_3 =$	1.196 + 0.700i	$B_3 =$	1.069 + 0.544i
	$A_4 =$	1.196 − 0.700i	$B_4 =$	1.330 + 0.000i
	$A_5 =$	0.937 − 0.381i	$B_5 =$	0.832 + 0.331i
	$A_6 =$	0.937 + 0.381i	$B_6 =$	0.832 − 0.331i
	$A_7 =$	2.205 + 0.404i	$B_7 =$	1.793 + 0.528i
	$A_8 =$	2.205 − 0.404i	$B_8 =$	1.793 − 0.528i
	$A_9 =$	1.666 + 0.475i	$B_9 =$	1.683 + 0.810i
	$A_{10} =$	1.666 − 0.475i	$B_{10} =$	1.683 − 0.810i

It should be noted that according to Aizikovich and Aleksandrov (1982) the exponential decrease with respect to γ for the layered coatings is characteristic of the kernel transform

$$L(\gamma) = 1 + \tfrac{1}{2}e^{-2\gamma h_1} + O\left(e^{-2\gamma h_1}\right), \quad \gamma \to \infty$$

where h_1 is the thickness of the upper layer. At the same time, the following type of behaviour at infinity is characteristic of blended coatings

$$L(\gamma) = 1 + C_1\gamma^{-1} + C_2\gamma^{-2} + O(\gamma^{-3}), \quad \gamma \to \infty$$

where C_1, C_2 are constants. Hence, the kernel transform for the layered coating (when $\gamma \to \infty$) is approaching to 1 more rapidly than in the case of the functional graded coating.

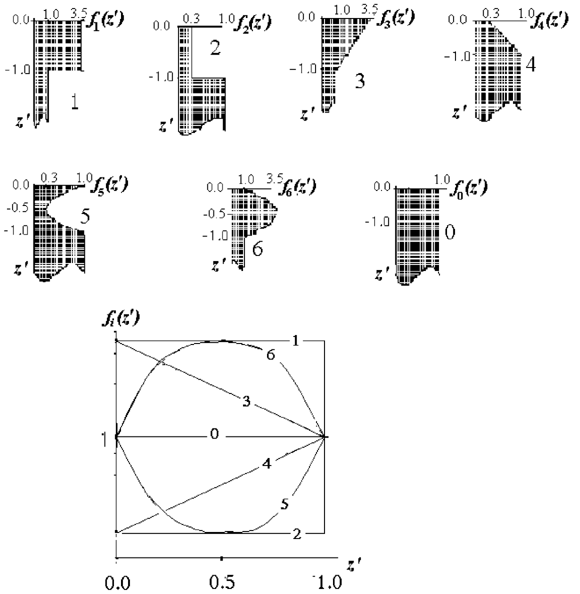

Fig. 2. Non-homogeneity laws describing considered variations of elastic modulus with depth, $f_i(z')$, $i = 0$–6.

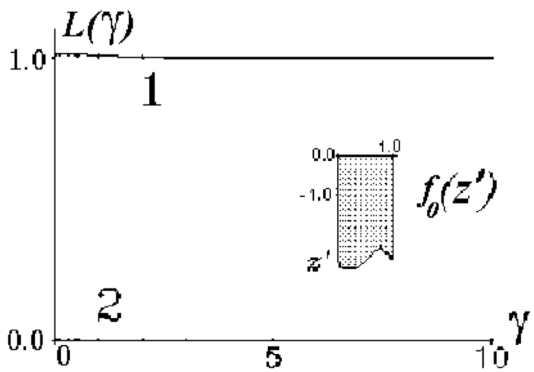

Fig. 3. Kernel transform $L(\gamma)$ of integral equation for non-homogeneity law $f_0(z')$.

It should be noted that the kernel transform for the two-layered law of non-homogeneity is approximated more exactly by the analytical expression (35).

6. The asymptotic solution of an integral equation of the problem

6.1. Existence and uniqueness of the solution of the integral equation of the problem for $L(\gamma)$ of class $\mathbf{\Pi}_N$

Eq. (29) can be written in terms of the operator for $L(\gamma)$ of class $\mathbf{\Pi}_N$ in the form

$$\mathbf{\Pi}_N t = f$$

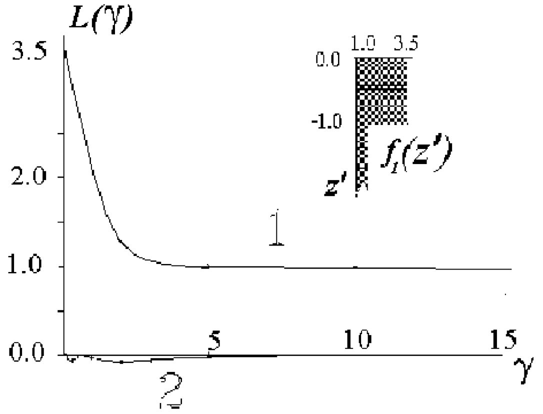

Fig. 4. Kernel transform $L(\gamma)$ of integral equation for non-homogeneity law $f_1(z')$.

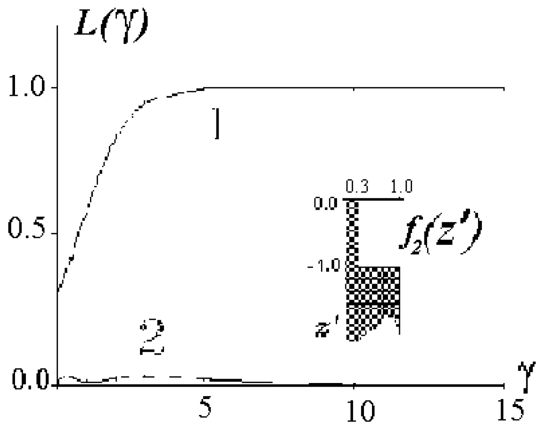

Fig. 5. Kernel transform $L(\gamma)$ of integral equation for non-homogeneity law $f_2(z')$.

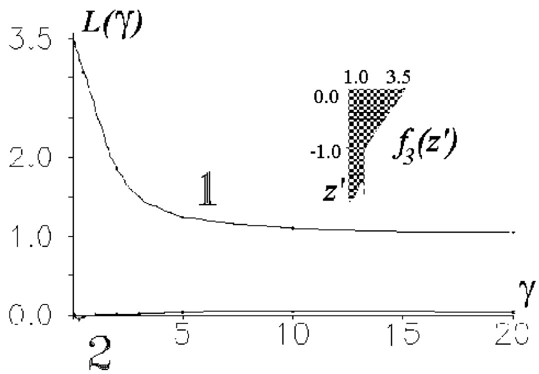

Fig. 6. Kernel transform $L(\gamma)$ of integral equation for non-homogeneity law $f_3(z')$.

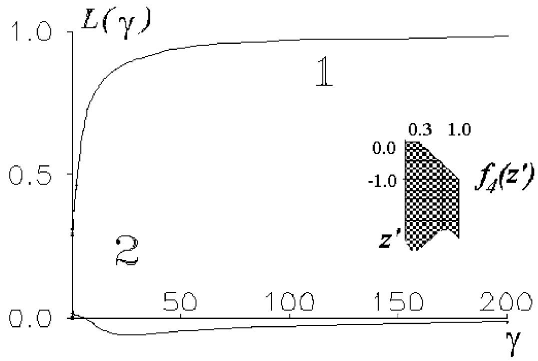

Fig. 7. Kernel transform $L(\gamma)$ of integral equation for non-homogeneity law $f_4(z')$.

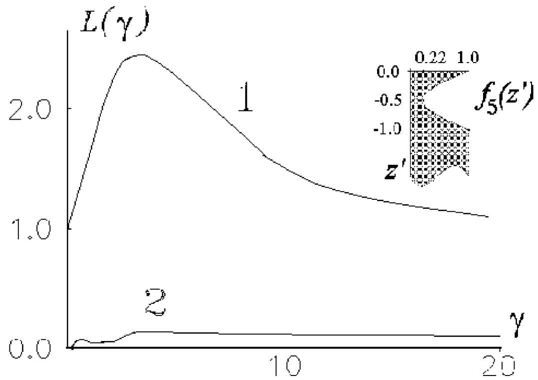

Fig. 8. Kernel transform $L(\gamma)$ of integral equation for non-homogeneity law $f_5(z')$.

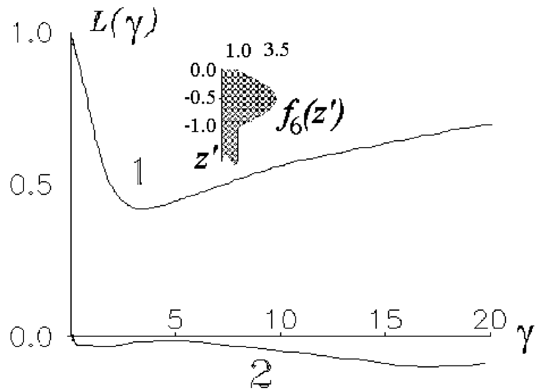

Fig. 9. Kernel transform $L(\gamma)$ of integral equation for non-homogeneity law $f_6(z')$.

Lemma 2. *Eq.* (29) *is solvable uniquely for* $L(\gamma)$ *of class* $\mathbf{\Pi}_N$ *in the class of functions* $\mathbf{C}(-1,1)$, *hence, the estimate*

$$\|t(r)\|_{\mathbf{C}(-1;1)} \leqslant m(\mathbf{\Pi}_N)\|f\|_{\mathbf{C}(-1;1)}, \quad m(\mathbf{\Pi}_N) = \text{const} \tag{45}$$

holds in $\mathbf{C}(-1,1)$. *Below,* $m(\mathbf{A})$ *shall denote a certain constant dependent on the specific form of the functions belonging to the class* $\mathbf{A}$.

Using the operators

$$\mathbf{U}_1^\zeta \varphi(r) = \frac{\mathrm{d}}{\mathrm{d}\zeta}\int_0^\zeta \frac{r}{\sqrt{\zeta^2 - r^2}}\varphi(r)\,\mathrm{d}r; \quad \left(\mathbf{U}_1^\zeta J_0(\alpha r) = \cos\alpha\zeta\right)$$

$$\mathbf{U}_2^\zeta \varphi(r) = \int_\zeta^\infty \frac{r}{\sqrt{\zeta^2 - r^2}}\varphi(r)\,\mathrm{d}r; \quad \left(\mathbf{U}_2^\zeta J_0(\alpha r) = \alpha^{-1}\ \cos\alpha\zeta\right)$$

we represent (29) in the form

$$\begin{cases} \int_0^\infty T(\alpha)L(\lambda\alpha)\cos\alpha\zeta\,\mathrm{d}\alpha = \Theta(0)g(\zeta), & 0 \leqslant \zeta \leqslant 1 \\ \int_0^\infty T(\alpha)\cos\alpha\zeta\,\mathrm{d}\alpha = 0, & \zeta > 1 \end{cases} \tag{46}$$

$$g(\zeta) = \mathbf{U}_1^\zeta f(r)$$

In our case

$$\mathbf{U}_1^\zeta\left[\delta - \frac{\psi(ra)}{a}\right] = \mathbf{U}_1^\zeta\left[\delta - \frac{ar^2}{2R}\right] = \delta - \frac{a}{R}\zeta^2 \tag{47}$$

We use the definition

$$\Theta(0)g(\zeta) = k_1 g_1(\zeta), \qquad g_1(\zeta) = \zeta^2 + \frac{k_2}{k_1}$$

From (47), $k_1 = -(a/R)\Theta(0)$, $k_2 = \delta\Theta(0)$.

In the case when

$$L(\lambda\alpha) = L_N(\lambda\alpha) = \frac{P_1(\lambda^2\alpha^2)}{P^2(\lambda^2\alpha^2)} = \prod_{i=1}^{N}\frac{\lambda^2\alpha^2 + A_i^2}{\lambda^2\alpha^2 + B_i^2} \tag{48}$$

by using the method developed by Aleksandrov (1973), we have obtained a solution of the problem. L_N has the representation:

$$L_N(\lambda\alpha) = 1 + \sum_{k=1}^{N} s_k B_k^2 \lambda^{-2}\left(\alpha^2 + B_k^2\lambda^{-2}\right)^{-1}$$

here $s_k = -L_N^{\tilde{k}}(iB_k\lambda^{-1})(1 - A_k^2 B_k^{-2})$. We use the definitions

$$L_N^{\tilde{k}}(\lambda\alpha) = \prod_{i=1,\tilde{k}}^{N}\frac{\lambda^2\alpha^2 + A_i^2}{\lambda^2\alpha^2 + B_i^2}$$

where $\prod_{i=1,k}^{N}$ means that the mth factor is absent in this product.

Let us introduce the function

$$p(\zeta) = \frac{1}{k_1}\int_0^\infty T(\alpha)\cos\alpha\zeta\,\mathrm{d}\alpha \tag{49}$$

Then, using the operational calculus methods (Lure, 1955), (46) is represented in the form

$$P_1(-D)p(\zeta) = P_2(-D)g_1(\zeta), \quad D = \frac{\mathrm{d}^2}{\mathrm{d}\zeta^2}, \quad \zeta \in [0;1] \tag{50}$$

with polynomials P_1 and P_2 defined in Eq. (48).

The solution of differential equation (50) can be written in the form

$$p(\zeta) = \sum_{i=1}^{N} C_i \cosh\left(A_i\lambda^{-1}\zeta\right) + L_N^{-1}(0)\left[\zeta^2 + \frac{k_2}{k_1} + 2\lambda^2\sum_{i=1}^{N}\left(A_i^{-2} - B_i^{-2}\right)\right] \tag{51}$$

where coefficients C_i, $i = 1,\ldots,N$ are unknown and will be defined later from the condition that this solution should satisfy identically the integral equation (46).

If we use an inverse Fourier transform in (49), then we obtain

$$T(\alpha) = \frac{2}{\pi}k_1\left\{L_N^{-1}(0)\left[2\frac{\alpha\cos\alpha - \sin\alpha}{\alpha^3} + \left(1 + \frac{k_2}{k_1} + 2S_1\right)\frac{\sin\alpha}{\alpha}\right]\right.$$
$$\left. + \sum_{i=1}^{N} C_i \frac{A_i\lambda^{-1}\sinh\left(A_i\lambda^{-1}\right)\cos\alpha + \alpha\sin\alpha\cosh\left(A_i\lambda^{-1}\right)}{\alpha^2 + A_i^2\lambda^{-2}}\right\} \tag{52}$$

where

$$S_1 = \lambda^2\sum_{i=1}^{N}\left(A_i^{-2} - B_i^{-2}\right)$$

According to (30) we obtain an expression for stresses, if we use the Hankel transform

$$t(r) = \int_0^\infty T(\alpha)J_0(\alpha r)\,\mathrm{d}\alpha \tag{53}$$

Substituting (52) in (53), we obtain

$$t(r) = \frac{2}{\pi}k_1\left\{L_N^{-1}(0)\left[-2\sqrt{1-r^2} + \left(1 + \frac{k_2}{k_1} + 2S_1\right)\frac{1}{\sqrt{1-r^2}}\right]\right.$$
$$\left. + \sum_{i=1}^{N} C_i\left(\frac{\cosh\left(A_i\lambda^{-1}\right)}{\sqrt{1-r^2}} - A_i\lambda^{-1}\int_r^1 \frac{\sinh\left(A_i\lambda^{-1}\zeta\right)}{\sqrt{\zeta^2 - r^2}}\,\mathrm{d}\zeta\right)\right\}, \quad 0 \leqslant r \leqslant 1 \tag{54}$$

We can find a constant k_2 from the condition $q(1) = 0$, which eliminates the diverging rams in Eq. (54):

$$L_N^{-1}(0)\left[-\frac{\delta R}{a} + 1 + 2S_1\right] + \sum_{i=1}^{N} C_i\cosh\left(A_i\lambda^{-1}\right) = 0 \tag{55}$$

Coefficients C_i are found from the condition that if we substitute $T(\alpha)$ from (52) in (46) then the last one must be identically satisfied for $L(\lambda\alpha)$ in the form (48).

$$\int_0^\infty T(\alpha)\left(1+\sum_{k=1}^N \frac{s_k B_k^2 \lambda^{-2}}{\alpha^2+B_k^2\lambda^{-2}}\right)\cos\alpha\zeta\,\mathrm{d}\alpha = k_1\zeta^2+k_2, \quad 0\leqslant\zeta\leqslant 1 \tag{56}$$

From (56) we obtain the linear algebraic system of equations for the coefficients C_i

$$\sum_{i=1}^N C_i \frac{A_i\lambda^{-1}\sinh\left(A_i\lambda^{-1}\right)\cos\alpha + B_k\lambda^{-1}\cosh\left(A_i\lambda^{-1}\right)}{B_k^2\lambda^{-2}-A_i^2\lambda^{-2}} + \left[L_N(0)B_k\lambda^{-1}\right]^{-1}\left\{-\frac{\delta R}{a}+1+2\left[\frac{B_k\lambda^{-1}+1}{B_k^2\lambda^{-2}} + \lambda^2\sum_{i=1}^N\left(A_i^{-2}-B_i^{-2}\right)\right]\right\} = 0, \quad 0\leqslant r\leqslant 1 \tag{57}$$

The system (57) is solvable uniquely if A_i, B_k satisfy conditions (35). The assertion of the Lemma 2 and the estimate (45) results from here. □

In accordance with (55), the expression for the normal contact stresses is represented in the form

$$t(r) = \frac{2}{\pi}\Theta(0)\frac{a}{R}\left\{2L_N^{-1}(0)\sqrt{1-r^2} + \sum_{i=1}^N C_iA_i\lambda^{-1}\int_r^1 \frac{\sinh\left(A_i\lambda^{-1}\zeta\right)}{\sqrt{\zeta^2-r^2}}\,\mathrm{d}\zeta\right\}, \quad 0\leqslant r\leqslant 1 \tag{58}$$

The acting force P is defined from the equilibrium condition of the indenter

$$P = 2\pi a^2\int_0^1 t(r)r\,\mathrm{d}r$$

Hence

$$P = \frac{4a^3}{R}\Theta(0)\left\{\frac{2}{3}L_N^{-1}(0) + \sum_{i=1}^N C_iA_i\lambda^{-1}\left(-\cosh\left(A_i\lambda^{-1}\right) + A_i\lambda^{-1}\sinh\left(A_i\lambda^{-1}\right)\right)\right\} \tag{59}$$

Theorem 2. *If the conditions* (35) *hold and the non-homogeneity law is such that* $\partial P/\partial a$, *being the function of* A_i, C_i *satisfies*

$$\frac{\partial P}{\partial a} > 0, \quad P\neq 0,\ a\neq 0 \tag{60}$$

then the operator $\mathbf{\Pi}_N$ *is reversible and the following estimate holds*

$$\|t(r)\|_{C(-1;1)} \leqslant \left\|\mathbf{\Pi}_N^{-1}\right\|\cdot\|f\|_{C(-1;1)} \tag{61}$$

Let us consider an example. For $N=1$ constants C_1 and can be defined from the next linear algebraic system of equations

$$\left(\frac{B_1}{A_1}\right)^2\left\{-\frac{\delta R}{a}+1+2\lambda^2(A_1^2-B_1^2)\right\} + C_1\cosh(A_1\lambda^{-1}) = 0$$

$$C_1\frac{A_1\sinh(A_1\lambda^{-1}) + B_1\cosh(A_1\lambda^{-1})}{B_1^2-A_1^2} + \frac{B_1}{A_1^2}\left\{-\frac{\delta R}{a}+1+2\lambda^2\left[\frac{B_1\lambda^{-1}+1}{B_1^2} + (A_1^{-2}-B_1^{-2})\right]\right\} = 0, \quad 0\leqslant r\leqslant 1$$

$$t(r) = \frac{2}{\pi}\Theta(0)\frac{a}{R}\left\{2\frac{B_1^2}{A_1^2}\sqrt{1-r^2} + C_1A_1\lambda^{-1}\int_r^1 \frac{\sinh(A_1\lambda^{-1}\zeta)}{\sqrt{\zeta^2-r^2}}\,d\zeta\right\}, \quad 0 \leqslant r \leqslant 1$$

6.2. Existence and uniqueness of the solution of the problem for $L(\gamma)$ of class $\mathbf{S}_{N,M}$

The Eq. (29) can be written in the operator form for $L(\gamma)$ of class $\mathbf{S}_{N,M}$ in the form

$$\mathbf{\Pi}_N t + \mathbf{\Sigma}_M t = f \tag{62}$$

Definition 4. We shall say that function $f(x)$, absolutely integrated on a segment $[0;1]$, satisfies condition $\mathbf{M}_0$, if a Fourier–Bessel expansion holds (M_f^0 is a certain constant), Koshliakov et al., 1962

$$f(x) = \sum_{n=1}^{\infty} a_n^0 J_0(\mu_n x), \sum_{n=1}^{\infty} | a_n^0 \mu_n | \leqslant M_f^0 < \infty$$

Here $\mu_1, \mu_2, \dots, \mu_k, \dots$ are positive roots of Bessel's function $J_0(x)$ indexed in increasing order, a_n^0, $n = 1, 2, \dots$ are coefficients of a Fourier–Bessel series expansion of $f(x)$.

$$a_n^0 = \frac{2}{J_1^2(\mu_n)}\int_0^1 tf(t)J_0(\mu_n t)\,dt$$

Lemma 3. *Operator* $\mathbf{\Pi}_N^{-1}\mathbf{\Sigma}_M$ *of the problem is a compression operator in space* $\mathbf{C}(-1,1)$ *when the condition* (35) *is satisfied, if* $0 < \lambda < \lambda^*$ *or* $\lambda > \lambda^0$, *where* λ^* *and* λ^0 *are certain fixed values of* λ.

We consider the operator $\mathbf{\Sigma}_M(t)$. Without loss of generality, we set $M = 1$ and thus obtaining

$$\mathbf{\Sigma}_1(t) = \int_0^1 t(\rho)\rho\,d\rho\int_0^{\infty}\frac{d_1\lambda^{-1}\alpha}{\alpha^2 + d_2^2\lambda^{-2}}J_0(\alpha\rho)\,d\alpha \tag{63}$$

where d_1, d_2 are the certain constants.

We represent $\mathbf{\Sigma}_1(t)$ as the Fourier–Bessel series

$$\mathbf{\Sigma}_1(t) = \sum_{k=1}^{\infty} a_k J_0(\mu_k r)$$

Coefficients a_k are found from the following formula

$$a_k = \frac{2d_1\lambda^{-1}}{J_1^2(\mu_k)(\mu_k^2 + d_2^2\lambda^{-2})}\left[\int_0^1 t(\rho)\rho J_0(\rho\mu_k)\,d\rho - \mu_k J_1(\mu_k)K_0(d_2\lambda^{-1})\int_0^1 t(\rho)\rho I_0(\rho d_2\lambda^{-1})\,d\rho\right] \tag{64}$$

where K_0 is the McDonald's function of the zero order, I_0 is the Bessel function of imaginary argument of the zero order.

Using the asymptotic estimates of cylindrical functions of an imaginary argument (Gradshtein and Ryzhik, 1965):

$$\begin{array}{ll} K_0(z) \sim -\ln z, I_0(z)' \sim 1 & \text{for } z \to 0 \\ K_0(z) \sim -e^{-z}\sqrt{\pi/2z}, I_0(z)' \sim e^z/\sqrt{2\pi z} & \text{for } z \to \infty \end{array} K_0$$

we obtain the following estimate from (64)

$$\begin{aligned} \max_{r\in[0;1]}|\mathbf{\Sigma}_1(t)| &\leqslant C_1\sum_{k=1}^{\infty}|a_k| \leqslant \lambda M^0, \quad \lambda\to 0(\lambda<\lambda^0) \\ \max_{r\in[0;1]}|\mathbf{\Sigma}_1(t)| &\leqslant C_2\sum_{k=1}^{\infty}|a_k| \leqslant \lambda^{-1} M^*, \quad \lambda\to \infty(\lambda>\lambda^*) \end{aligned} \tag{65}$$

here C_1, C_2 are the certain constants and constants M^0 and M^* are independent of λ.

Hence, analogously to the estimates of Lemma 2, λ can be selected so that operator $\mathbf{\Pi}_N^{-1}\mathbf{\Sigma}_M$ will be a compression operator under the conditions of Lemma 2, Kantorovich and Akilov (1982). On this basis, by applying the Banach principle of compressed mappings to the Eq. (62)

$$t + \mathbf{\Pi}_N^{-1}\mathbf{\Sigma}_M t = \mathbf{\Pi}_N^{-1} f \tag{66}$$

we obtain the proof of the existence and uniqueness of the solution of (29) under the constraints imposed by Lemma 2. □

That means the following estimates take place.

Theorem 3. *The Eq.* (29) *of the problem is solvable uniquely in the space* $\mathbf{C}(-1,1)$ *for* $L(\gamma)$ *of the class* $\mathbf{S}_{N,M}$, if $f(r)$ *is even function and satisfies the condition* $\mathbf{M}_0$ *for* $0<\lambda<\lambda^*$ *or* $\lambda>\lambda^0$, *where* λ^* *and* λ^0 *are certain fixed values of* λ. *Hence, the following estimate holds*:

$$\|t(r)\|_{\mathbf{C}(-1;1)} \leqslant m(\mathbf{\Pi}_N, \mathbf{\Sigma}_M) M_f^0$$

Finally, we formulate the following theorem:

Theorem 4. *The Eq.* (29) *of the problem is solvable uniquely in the space* $\mathbf{C}(-1,1)$ *for* $L(\gamma)$ *of the class* $\mathbf{S}_{N,\infty}$ *if* $f(r)$ *is even function and satisfies the condition* $\mathbf{M}_0$ *for* $0<\lambda<\lambda^*$ *or* $\lambda>\lambda^0$, *where* λ^* *and* λ^0 *are the certain fixed values of* λ, *hence, the following estimate holds*:

$$\|t(r)\|_{\mathbf{C}(-1;1)} \leqslant m(\mathbf{\Pi}_N, \mathbf{\Sigma}_\infty) M_f^0 \tag{67}$$

The proof of Theorem 4 follows from the assertions in Theorem 2 and 3 and is analogous to the one carried out in Babeshko (1971, 1972).

6.3. The stiffness concept of the depthwise non-homogeneous material

As a result of the penetration of the indenter into non-homogeneous material we can obtain the relation between the impressing force and the displacement of the indenter. Directly using this relation is not convenient for the determination of coating non-homogeneity. We define an expression which is referred to as the stiffness of the material

$$S = \frac{3}{4}\frac{P}{a\chi}\frac{1}{1-v^2}$$

where a is the contact zone radius, χ is the displacement of the indenter, v is the Poisson's ratio. For the homogeneous material the stiffness is a constant equivalent to the shear modulus of the foundation (Johnson, 1985). For the non-homogeneous material, $S(a)$ is a function depending on the contact zone size.

6.4. Numerical results

Fig. 10 shows the graphs of the quantity $q_0(r)$, which is the distribution of the contact stresses under a non-deformable spherical indenter for a homogeneous half-space with $E(z) = E_0$.

Figs. 11–16 show graphs of the ratio $\mathbf{\Phi}_i(r) = q_i(r)q_0^{-1}(r)$ which characterizes the distribution of normal contact stresses $q_i(r)$ under a non-deformable spherical indenter, for the non-homogeneity of type $f_i(z)$ $(i = 1, \ldots, 6)$ from the relation (44). The values of $q_i(r)$ were found by the formula (58) with $N = 10$. The ratio of $q_i(r)$ to $q_0(r)$ is considered for equal values of the contact zones for a spherical indenter impressed into a non-homogeneous and, respectively, into a homogeneous half-space the elasticity modulus of which is equal to the elasticity modulus of the substrate.

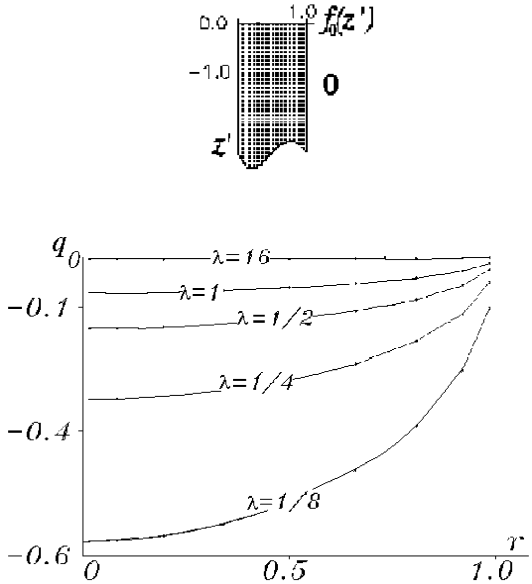

Fig. 10. Distribution of contract stresses under a non-deformable spherical indenter for a homogeneous half-space.

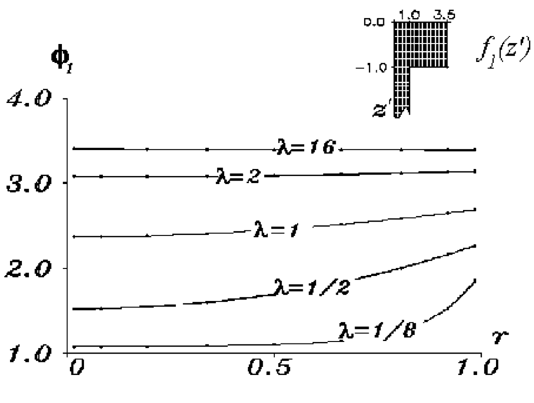

Fig. 11. Graph of the ratio $\mathbf{\Phi}_1(r) = q_1(r)/q_0(r)$ which characterizes the distribution of normal contact stresses $q_1(r)$ under a non-deformable spherical indenter.

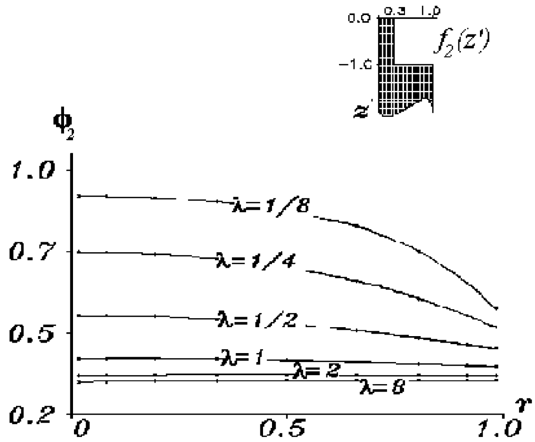

Fig. 12. Graph of the ratio $\mathbf{\Phi}_2(r) = q_2(r)/q_0(r)$ which characterizes the distribution of normal contact stresses $q_2(r)$ under a non-deformable spherical indenter.

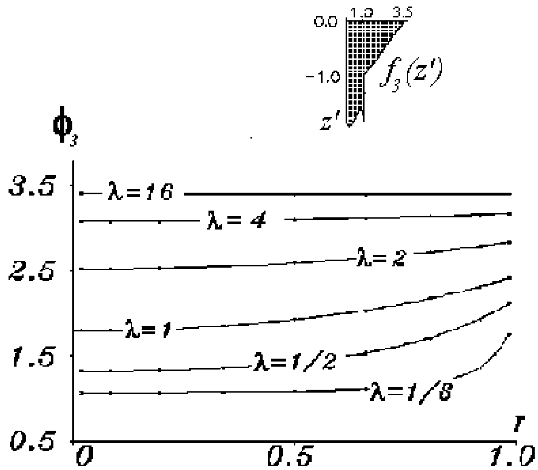

Fig. 13. Graph of the ratio $\mathbf{\Phi}_3(r) = q_3(r)/q_0(r)$ which characterizes the distribution of normal contact stresses $q_3(r)$ under a non-deformable spherical indenter.

Fig. 17 shows graphs of $S(\lambda^{-1})/S_0$ the ratio of the stiffness of the non-homogeneous coatings S to the stiffness of the substrate—a homogeneous half-space S_0 for the five cases of the non-homogeneity laws mentioned above $f_i(z')$, $(i = 0, 1, 2, 3, 4)$. To make the graph more descriptive we present them using logarithmic scale. The curve numbers corresponds to the variation laws of the elasticity modulus. Fig. 17 shows that using the results of non-destructive indentation experiments we can evaluate the variation of the elasticity modulus with the depth. Moreover, it shows that this kind of test gives the possibility to distinguish changes of surface layer properties not only in terms of more soft or more hard, but also to determine the variation of the coatings elastic properties with the depth (blended or layered), provided that the layer size is known.

In Fig. 18, curves 0, 5 and 6 show the ratio of the stiffness of non-homogeneous coatings, S, to the stiffness of substrate, S_0, for $f_i(z')$, $(i = 0, 5, 6)$ which characterizes the non-homogeneity laws described above.

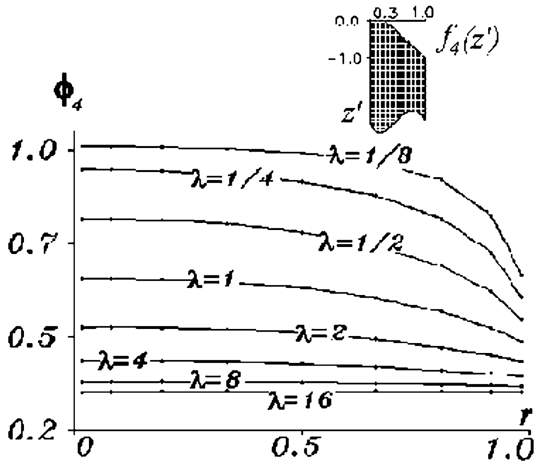

Fig. 14. Graph of the ratio $\Phi_4(r) = q_4(r)/q_0(r)$ which characterizes the distribution of normal contact stresses $q_4(r)$ under a non-deformable spherical indenter.

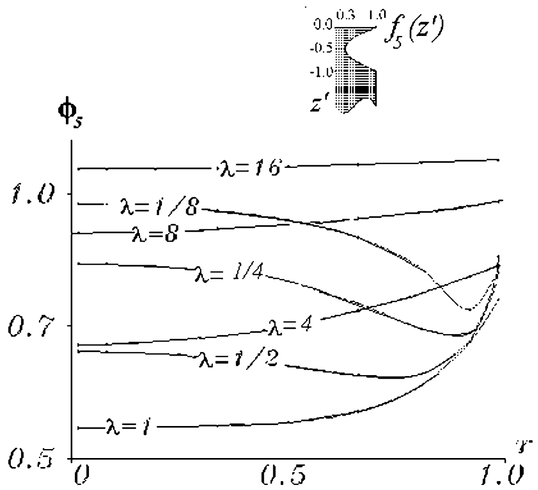

Fig. 15. Graph of the ratio $\Phi_5(r) = q_5(r)/q_0(r)$ which characterizes the distribution of normal contact stresses $q_5(r)$ under a non-deformable spherical indenter.

The curve numbers correspond to the variation laws of elastic modulus. It is obvious from Figs. 17 and 18 that the middle region of the stiffness change graph is the most informative one $(1/4H \leqslant a \leqslant 4H)$.

7. A posteriori accuracy evaluation of the bilateral asymptotically exact method of the solution of integral equations

It was shown above that the problem of determining contact stress distribution arising from indentation of a spherical indenter into the depthwise non-homogeneous layer coupled with a homogeneous half-space

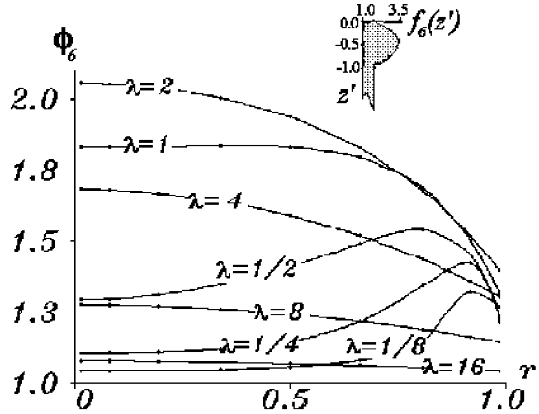

Fig. 16. Graph of the ratio $\mathbf{\Phi}_6(r) = q_6(r)/q_0(r)$ which characterizes the distribution of normal contact stresses $q_6(r)$ under a non-deformable spherical indenter.

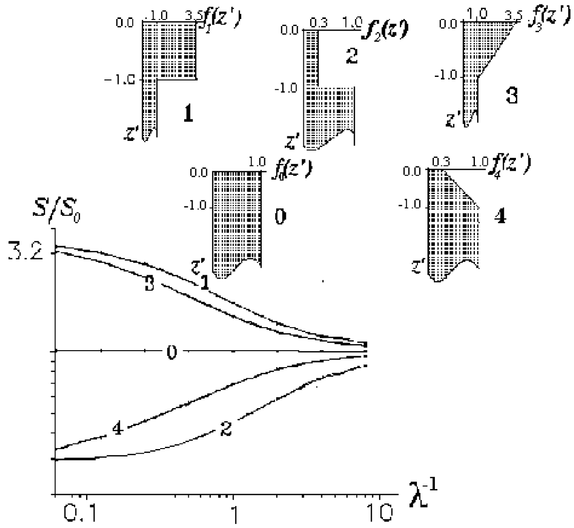

Fig. 17. Graph of the ratio $S(\lambda^{-1})/S_0$ for the $f_i(z')$, $i = 0$–4.

is reduced to the solution of the dual integral equation in the form (29) with the additional condition of indenter equilibrium

$$P = 2\pi \int_0^1 q(\rho)\rho \, d\rho$$

Constructing the approximate bilateral asymptotically exact solution of the problem, we substituted the kernel of (29) by its approximation in form (35).

After determining the contact stresses $q_N(\rho)$ distribution, we can find the integral characteristic of the displacement error under the indenter

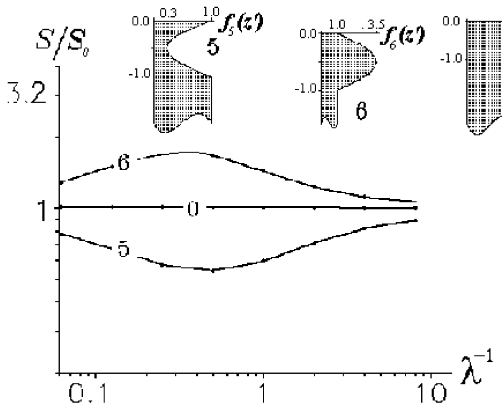

Fig. 18. Graph of the ratio $S(\lambda^{-1})/S_0$ for the $f_i(z')$, $i = 0, 5, 6$.

$$\Delta_N(r) = \int_0^1 q_N(\rho)\rho\,\mathrm{d}\rho \int_0^\infty (L(\lambda\alpha) - L_N(\lambda\alpha))J_0(\alpha\rho)J_0(\alpha r)\,\mathrm{d}r \tag{68}$$

which is obtained as a result of the substitution of the kernel transform by its approximation. The function $f(r)$ from the right part of (29) corresponds to surface displacement values of the non-homogeneous half-space under the spherical indenter and so the physical sense of the error is the relative displacement. The integral expression in Eq. (68) is bounded, hence, we can easily find $\Delta_N(r)$ with the help of quadrature formulas (Gauss formula, for example). The $\Delta_N(r)$ value gives the absolute error for the indenter displacement determination. We obtain the relative error rate as the ratio:

$$\varepsilon(r) = 100 \times \frac{\Delta_N(r)}{w(r)}$$

where $w(r)$ is the surface displacement of a half-space under the indenter.

7.1. The numerical investigations of the solution error depending on λ ($\lambda \to 0, \lambda \to \infty$)

The results of the calculations of the value $\varepsilon(r)$ for different non-homogeneity laws are shown on Figs. 19–25. In the top part of the figures the relation of the elasticity modulus change with layer depth is graphed. Graphs 19–25 reveal the correlation of the errors and the different non-homogeneity laws.

When we construct the approximations $L_N(\gamma)$ we can vary both the number of terms in the product (36) and the mapping parameter. This gives us the opportunity of controlling the approximation accuracy within certain limits. Thus, selecting the mapping parameter value as equal to the abscissa of the maximum error $(L(\gamma) - L_N(\gamma))/L(\gamma)$, we can reduce its value to some value linked with the elastic modulus change law (usually these values are achieved for a few points γ).

Fig. 26 shows the integral error of the displacement value $\omega(\lambda)$

$$\omega(\lambda) = \int_0^1 \varepsilon(r, \lambda)\,\mathrm{d}r$$

The figure reveals that the theoretically determined bilateral asymptotic properties of the solution $(\lambda \to 0, \lambda \to \infty)$ are confirmed by numerical experiments.

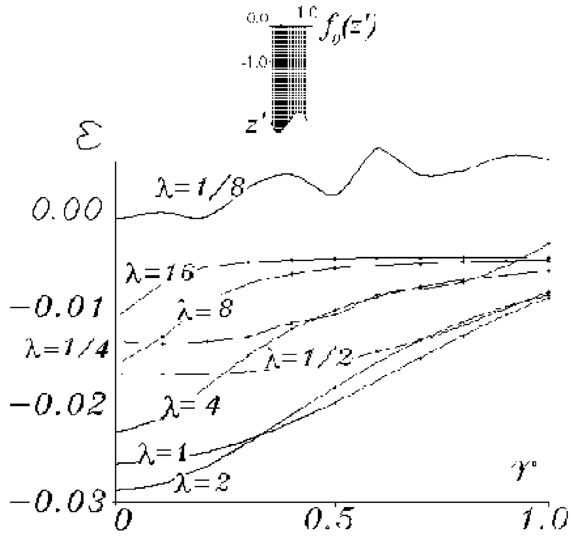

Fig. 19. Relative error $\varepsilon(r)$ of surface displacement determination for homogeneous half-space.

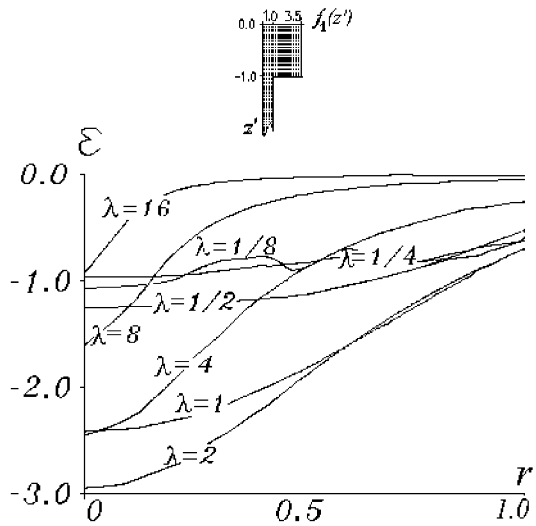

Fig. 20. Relative error $\varepsilon(r)$ of surface displacement determination for law of non-homogeneity $f_1(z') = 3.5$.

8. Conclusion

In this paper the Hertzian contact problem for both layered and functional graded half-space is studied analytically. We presumed that the variation with depth of the Lame coefficients in the half-space has general nature (arbitrary continuous or piecewise continuous functions of depth). We assumed elasticity

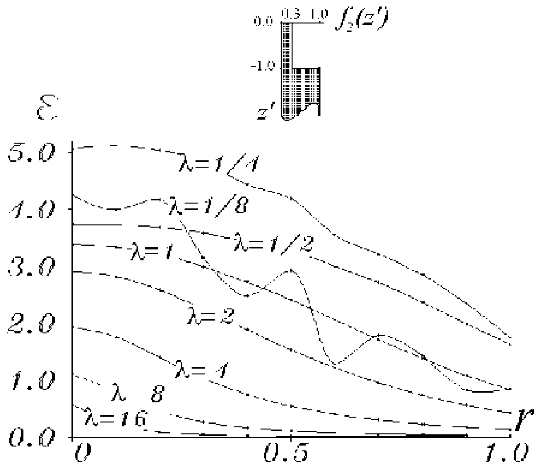

Fig. 21. Relative error $\varepsilon(r)$ of surface displacement determination for law of non-homogeneity $f_2(z') = 1/3.5$.

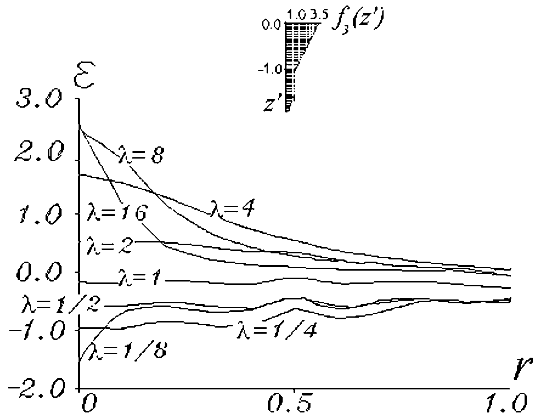

Fig. 22. Relative error $\varepsilon(r)$ of surface displacement determination for linear decreasing law of non-homogeneity.

properties of a half-space to become stable with depth, i.e., it was imagined as a non-homogeneous layer which is completely coupled with a homogeneous half-space. For the first stage we used the method of integral transforms for the solution of this problem. The two-point boundary problem for the system of ordinary differential equations with variable coefficients was solved using an effective special scheme in order to construct the transform of the integral equation kernel.

The numerically constructed kernel transform was approximated by an analytical expression of a special type, so that it has become possible to obtain a closed analytical solution of the approximate integral equation. The resulting approximate solution of the problem was shown to be the bilateral asymptotically exact one for small and large values of a characteristic geometric parameter.

Error evaluations for constructed approximate analytical solutions of the problem were carried out theoretically. The method can be easily implemented as the set of programs, which are PC compatible.

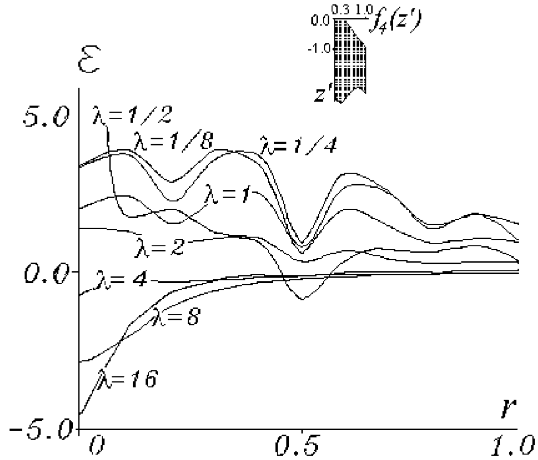

Fig. 23. Relative error $\varepsilon(r)$ of surface displacement determination for linear increasing law of non-homogeneity.

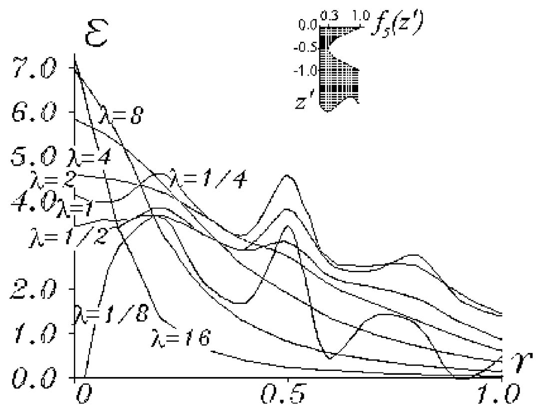

Fig. 24. Relative error $\varepsilon(r)$ of surface displacement determination for sinusoidal (concave) law of non-homogeneity.

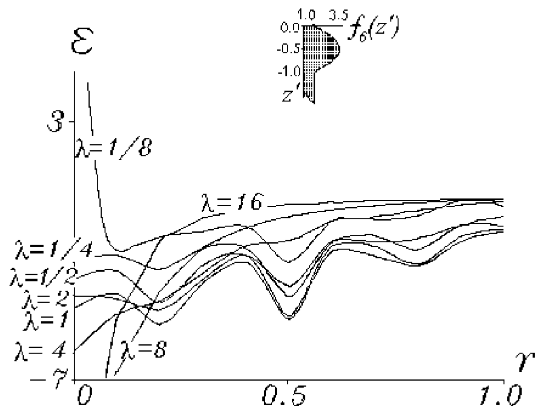

Fig. 25. Relative error $\varepsilon(r)$ of surface displacement determination for sinusoidal (convex) law of non-homogeneity.

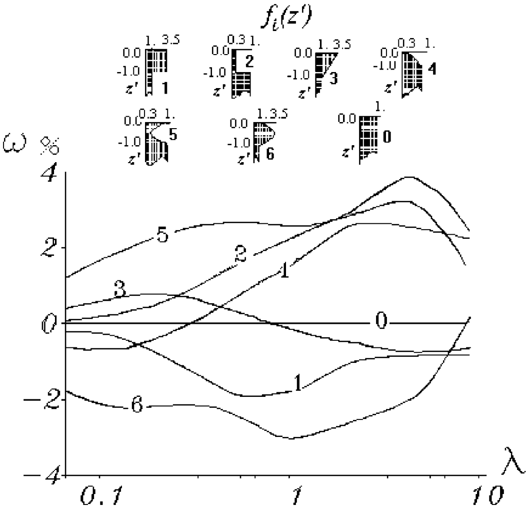

Fig. 26. Integral error of the displacement value $\omega(\lambda)$ determination.

Acknowledgements

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